Remarks:

- The (54)-(55) do not rely on perturbation theory, they are expressed in terms of $S = S_{full}$ (or $\mathcal{L} = \mathcal{L}_{full}$), do not assume $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$. Therefore, they can be at the base of non-perturbative ("lattice") calculations.
- From (54-55), once one has calculated $\langle 0|T\phi ..|0\rangle$, it's possible to extract in principle non-perturbative S-matrix elements via LSZ reduction.
- Sometime one absorbe the normalization $\int[D\phi]\exp\{iS\}$ into $[D\phi]$ so that $\int[D\phi]\exp\{iS\}\cdot 1 = \langle 0|0\rangle = 1$

Functional Methods

Like for the harmonic oscillator, it's useful to introduce the functional generator $Z[J]$ with general S

$$(56) \quad \boxed{Z[J] \equiv \int[D\phi]\exp(-S + J\cdot\phi)}$$

shorthand notation for $J\cdot\phi = \int d^4x\, J_i(x)\,\theta^i(x)$ or $J_\alpha \theta^\alpha$

both internal & spacetime index

Remark: it's a sort of Functional Fourier Transform

$$(57) \quad Z[J] = Z[0]\left(1 + \int d^4x\, J_\lambda(x)\langle\theta_\lambda(x)\rangle + \frac{1}{2!}\int d^4x_1 d^4x_2\, J_{\lambda_1}(x_1)\, J_{\lambda_2}(x_2)\langle T\theta_{\lambda_1}(x_1)\theta_{\lambda_2}(x_2)\rangle + \dots\right.$$

$$= Z[0]\left(\frac{1}{n!}J_{\alpha_1}\dots J_{\alpha_n}\langle T\theta_{\alpha_1}\dots\theta_{\alpha_n}\rangle\right)$$

so that any correlator is just the functional derivative w.r.t. J

(58) $$\boxed{\frac{\delta^n Z[J]}{\delta J_{\alpha_1}\dots\delta J_{\alpha_n}}\frac{1}{Z[J]}\Bigg|_{J=0} = \langle 0|\,T\,\mathcal{O}_{\alpha_1}\dots\mathcal{O}_{\alpha_n}\,|0\rangle}$$

(again, this is a compact notation for $\delta/\delta J_i(x)$ an $\mathcal{O}_{\alpha_i} = \mathcal{O}_i(x_i)$)

Remember that we are doing path integral from $t_i \to -\infty$ to $t_f \to +\infty$ so that (up to vacuum wavefunctionals $\langle \mathcal{O}(t_{i,f}\to\mp\infty, \vec{x})|0\rangle$) the $Z[J]$ is nothing but the vacuum-to-vacuum transition in the presence of external current J

(59) $$Z[J] \propto \langle 0|\;0\rangle_J$$

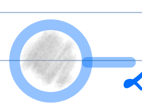
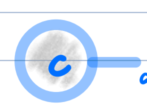
it's enough to know the vacuum-persistent amplitude $\forall J$, to be able to extract all correlators at $J=0$

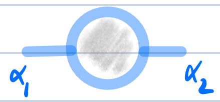
— Connected Correlators & their functional —

There is a more primitive notion of correlators: Connected Correlators

They are better behaved when the points are separated to infinity - they decay - and can reconstruct full correlator out of them. (Moreover, they will be directly linked to connect. amplitudes via (52))

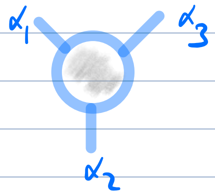
Let's give a sort of recursive definition:

(60)  $_{\alpha_1} = \langle \mathcal{O}_{\alpha_1}\rangle = \langle \mathcal{O}_{\alpha_1}\rangle_c =$  $_{\alpha_1}$ c: for connected

(61) 

$$= \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle = \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle_c + \langle \mathcal{O}_{\alpha_1} \rangle_c \langle \mathcal{O}_{\alpha_2} \rangle_c$$

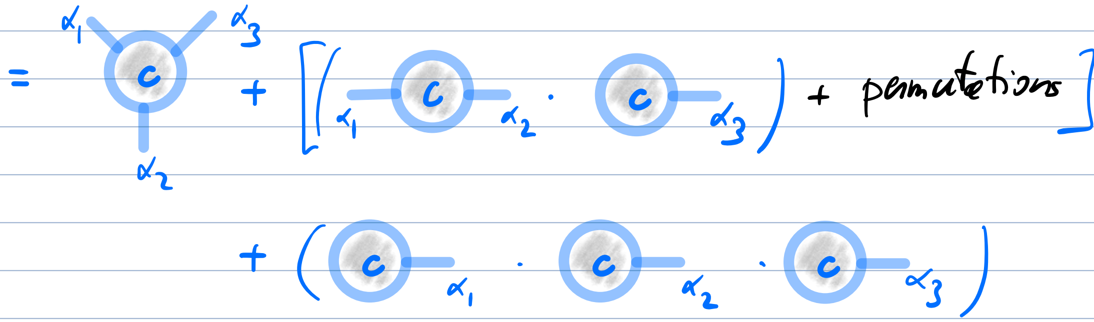
T-ordering left understood

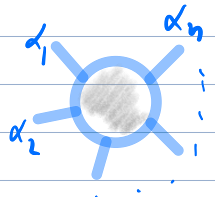
(62) 

$$= \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \mathcal{O}_{\alpha_3} \rangle = \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \mathcal{O}_{\alpha_3} \rangle_c$$

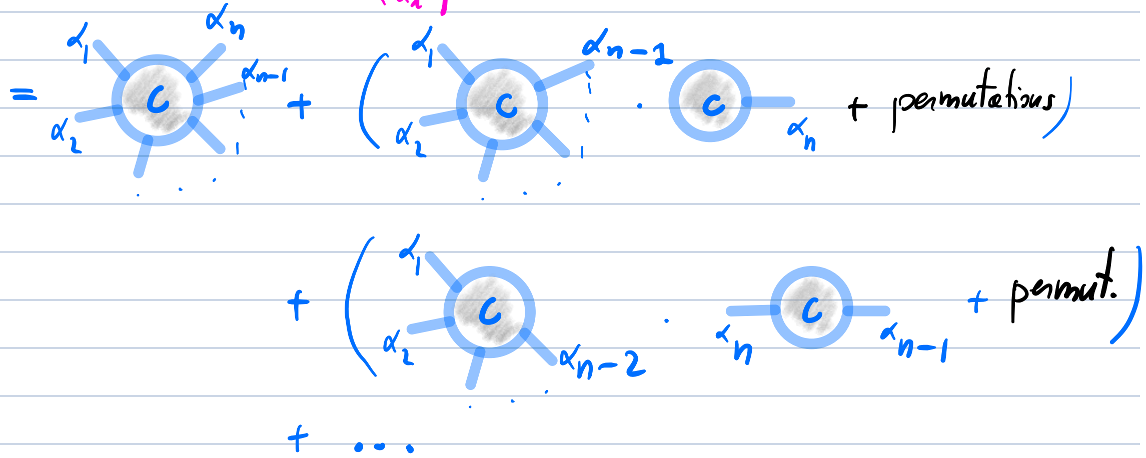
$$+ \left[\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle_c \langle \mathcal{O}_{\alpha_3} \rangle_c + \text{permutations} \right]$$

$$+ \langle \mathcal{O}_{\alpha_1} \rangle_c \langle \mathcal{O}_{\alpha_2} \rangle_c \langle \mathcal{O}_{\alpha_3} \rangle_c$$



(63) 

$$= \langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \rangle = \sum_{\substack{\text{all partitions} \\ \{\alpha_i\}}} \prod_{\text{parts}} \langle \ldots \rangle_c$$



Knowing connected $\longleftrightarrow$ Knowing correlator

Example $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle_c = \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle - \langle \mathcal{O}_{\alpha_1} \rangle_c \langle \mathcal{O}_{\alpha_2} \rangle_c$

$$= \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle - \langle \mathcal{O}_{\alpha_1} \rangle \langle \mathcal{O}_{\alpha_2} \rangle$$

(64) $$\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \mathcal{O}_3 \rangle_c = \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \mathcal{O}_{\alpha_3} \rangle - \Big(\underbrace{\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle_c}_{\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle - \langle \mathcal{O}_{\alpha_1} \rangle \langle \mathcal{O}_{\alpha_2} \rangle} \langle \mathcal{O}_{\alpha_3} \rangle_c + \text{perm} \Big) - \langle \mathcal{O}_{\alpha_1} \rangle_c \langle \mathcal{O}_{\alpha_2} \rangle_c \langle \mathcal{O}_{\alpha_3} \rangle_c$$

$\langle \mathcal{O}_{\alpha_i} \rangle$

$\vdots$

Connected correlators decay at large separation points, because of the cluster decomposition principle:

(65) $$\langle \mathcal{O}_{i_1}(x_1+a)\dots\mathcal{O}_{i_n}(x_n+a)\;\mathcal{O}_{j_1}(y_1)\dots\mathcal{O}_{j_m}(y_m)\rangle \xrightarrow[a\to\infty]{\text{cluster}} \langle\mathcal{O}_{\alpha_1}\dots\mathcal{O}_{\alpha_n}\rangle\langle\mathcal{O}_{\beta_1}\dots\mathcal{O}_{\beta_m}\rangle$$

(in Minkowski a is spacelike, in Euclidean it is automatically so)

(66) $$\Rightarrow \langle \mathcal{O}_{i_1}(x_1+a)\dots\mathcal{O}_{i_n}(x_n+a)\;\mathcal{O}_{j_1}(y_1)\dots\mathcal{O}_{j_m}(y_m)\rangle_c \xrightarrow{a\to\infty} 0$$

We can proceed recursively.

(67) $$\langle \mathcal{O}_{i_1}(x_1+a)\,\mathcal{O}_{i_2}(x_2)\rangle \xrightarrow{\text{cluster}} \langle\mathcal{O}_{i_1}(x_1)\rangle\langle\mathcal{O}_{i_2}(x_2)\rangle \underset{(61)}{\Rightarrow} \langle\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle \to 0$$

so the 2-pt is ok. Let's assume now n-point is ok & check what happens for n+1-point, in a particular case (only one cluster):

(68) $$\langle \mathcal{O}_{i_1}(x_1)\dots\mathcal{O}_{i_n}(x_n)\,\mathcal{O}_{i_{n+1}}(x_{n+1}+a)\rangle \xrightarrow{\text{cluster}} \langle\mathcal{O}_{i_1}(x_1)\dots\mathcal{O}_{i_n}(x_n)\rangle\langle\mathcal{O}_{\alpha_n}\rangle$$

$$\overset{\text{def.}}{=} \sum_{\substack{\text{all partition}\\ n\text{-points}\\ \text{and no } x_{n+1}}} \prod\langle\dots\rangle_c\,\langle\mathcal{O}_{\alpha_n}\rangle$$

it matches (68)

but on the other hand $$\langle\mathcal{O}_{\alpha_1}\dots\mathcal{O}_{\alpha_{n+1}}\rangle \overset{\text{def}}{=} \sum_{\substack{\text{partitions}\\ n\text{-point } \& \text{ no } x_{n+1}}}\prod\langle\ \rangle_c\langle\mathcal{O}_{\alpha_{n+1}}\rangle_c$$

$$+\sum_{\substack{\text{partition}\\ n\text{-points } \&\\ \text{no } x_j\neq x_{n+1}}}\prod\langle\dots\mathcal{O}_{i_{n+1}}(x_n+a)\dots\rangle_c\langle\mathcal{O}_j\rangle_c+\sum_{\substack{\text{partitions}\\ (n-1)\text{-points}}}\langle\ \rangle_c\langle\ \rangle_c+\dots$$

less than (n-1) pts

by assumption on n-point correlators (and lower) $\to 0$

$$+\langle\mathcal{O}_{\alpha_1}\dots\mathcal{O}_{\alpha_{n+1}}\rangle_c\to 0$$ since (68) it's already matched

Functional generator connected correlators : $W[J]$

is obtained by taking the log of $Z[J]$

$$(69) \qquad W[J] = \log Z[J] \qquad\qquad Z[J] = \exp\left(W[J]\right)$$

indeed :

$$(70) \qquad \frac{\delta}{\delta J_{\alpha_n}} \cdots \frac{\delta}{\delta J_{\alpha_2}} Z[J] = \frac{\delta}{\delta J_{\alpha_n}} \cdots \frac{\delta}{\delta J_{\alpha_2}} \left(e^{W[J]} \frac{\delta W}{\delta J_{\alpha_1}} \right) =$$

$$= \frac{\delta}{\delta J_{\alpha_n}} \cdots \frac{\delta}{\delta J_{\alpha_3}} e^{W[J]} \left(\frac{\delta^2 W}{\delta J_{\alpha_2} \delta J_{\alpha_1}} + \frac{\delta W}{\delta J_{\alpha_2}} \frac{\delta W}{\delta J_{\alpha_1}} \right)$$

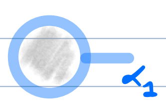
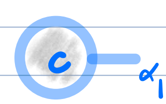
$$= \frac{\delta}{\delta J_{\alpha_n}} \cdots \frac{\delta}{\delta J_{\alpha_4}} e^{W[J]} \Bigg(\frac{\delta^3 W}{\delta J_{\alpha_3} \delta J_{\alpha_2} \delta J_{\alpha_1}} + \frac{\delta^2 W}{\delta J_{\alpha_3} \delta J_{\alpha_2}} \frac{\delta W}{\delta J_{\alpha_1}} + \frac{\delta W}{\delta J_{\alpha_3} \delta J_{\alpha_1}} \frac{\delta W}{\delta J_{\alpha_2}} +$$

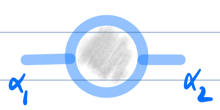
$$+ \frac{\delta W}{\delta J_{\alpha_2} \delta J_{\alpha_1}} \frac{\delta W}{\delta J_{\alpha_3}} + \frac{\delta W}{\delta J_{\alpha_3}} \frac{\delta W}{\delta J_{\alpha_2}} \frac{\delta W}{\delta J_{\alpha_1}} \Bigg)$$

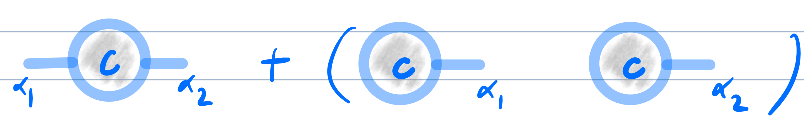
$$= \dots$$

$$(71) \qquad = \underset{\underset{Z[J]}{\parallel}}{e^{W[J]}} \Bigg(\frac{\delta^n W}{\delta J_{\alpha_n} \cdots \delta J_{\alpha_1}} + \left(\frac{\delta^{n-1} W}{\delta J_{\alpha_1} \cdots \delta J_{\alpha_{n-1}}} \frac{\delta W}{\delta J_{\alpha_n}} + \text{perm.} \right) + \left(\frac{\delta^{n-2} W}{\delta J_{\alpha_1} \cdots \delta J_{\alpha_{n-2}}} \frac{\delta^2 W}{\delta J_{\alpha_{n-1}} \delta J_{\alpha_n}} + \text{perm.} \right) + \dots \Bigg)$$

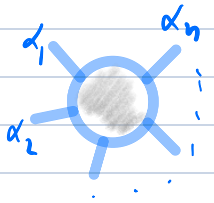
it has exactly the correct pattern of the connected correlators

(72)  $= \langle \vartheta_{\alpha_1} \rangle = \left.\frac{\delta W}{\delta J}\right|_{J=0} =$ 

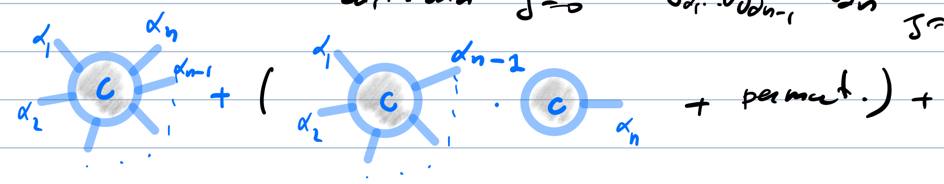
(73)  $= \langle \vartheta_{\alpha_1} \vartheta_{\alpha_2} \rangle = \langle \vartheta_{\alpha_1} \vartheta_{\alpha_2} \rangle_c + \langle \vartheta_{\alpha_1} \rangle_c \langle \vartheta_{\alpha_2} \rangle_c = \partial$

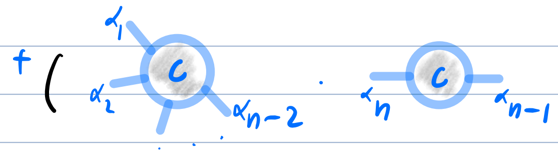
$=$ 

$$= \left. \delta^2 W / \delta J_{\alpha_1} \delta J_{\alpha_2} + \delta W / \delta J_{\alpha_1} \, \delta W / \delta J_{\alpha_2} \right|_{J=0}$$

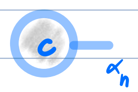
(74) 

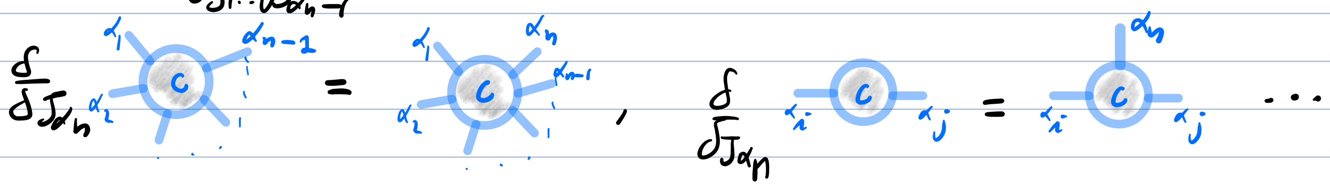
$$= \langle O_{\alpha_1} \ldots O_{\alpha_n} \rangle = \frac{\delta^n W}{\delta J_{\alpha_1} \ldots \delta J_{\alpha_n}} \Big|_{J=0} + \left(\frac{\delta^{n-1} W}{\delta J_{\alpha_1} \ldots \delta J_{\alpha_{n-1}}} \frac{\delta W}{\delta J_{\alpha_n}} \Big|_{J=0} + \text{perm.} \right) + \ldots$$

= 



This is clear: any time an extra $\frac{\delta}{\delta J(x)}$ acts, it either hit $e^W = Z[J]$ multiplying all previous sum of product of connected diagrams by  α_n

i.e. giving  α_n $\cdot \Big(\underset{\substack{\text{partition} \\ (n-1)\text{pts}}}{\sum \Pi} \langle \ \rangle_c \Big)$; or it hit a previous derivative $\frac{\delta^{n-1} W}{\delta J_1 \ldots \delta J_{\alpha_{n-1}}}$ and grow an extra leg, e.g.

(75) 

so to cover all ways of combining products of connected points to form the full disconnected amplitude.

Example: Free Scalar Theory

(76) $$S = \int d^4x \, \frac{1}{2} \phi \left[-\partial^2 + m^2 \right] \phi \qquad \text{Euclidean}$$

$\hookrightarrow$ $i S_{Mink} = - S_{Euc.} \, (t \to i t (1 - i\varepsilon))$

(77) $$Z[J] = \int [d\Phi] \exp(-S + J \cdot \phi)$$

$$S_{Mink} = \int d^4x \, \frac{\phi}{2} (-\partial^2 - m^2 + i\varepsilon) \phi$$

$$= \int [d\phi] \exp\left(-\tfrac{1}{2} \phi \, (-\partial^2 + m^2) \, \phi + J \cdot \phi \right) \underset{\substack{\text{Gaussian} \\ \text{Eq. 30-31}}}{\propto} \exp\Big(\tfrac{1}{2} J \cdot \underbrace{(-\partial^2 + m^2)^{-1}}_{\Delta} \cdot J \Big)$$

(78) $$\Delta_{xy} = \left[\delta^4(x-y) \left(-\partial_y^2 + m^2 \right) \right]^{-1} = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik(x-y)} \frac{1}{k^2 + m^2} \qquad \text{Feynman propagator}$$

$$\left(\int d^4y \, \delta^4(x-y) (-\partial^2 + m^2) \int \frac{d^4k}{(2\pi)^4} \, e^{-ik(y-z)} \frac{1}{k^2 + m^2} = \int d^4y \, \delta^4(x-y) \, \delta^4(y-z) = \delta^4(x-z) \quad \text{ok} \right)$$

(79) $W[J] = \frac{1}{2} J \cdot \Delta \cdot J = \int d^4z \int d^4y \, \frac{1}{2} J(y)\, \Delta(y-z)\, J(z) = \int \frac{d^4k}{(2\pi)^4} \hat{J}(-k) \frac{1}{k^2+m^2} \hat{J}(k)$

quadratic in the field theory

$\Downarrow$

(80) $\left.\frac{\delta W}{\delta J}\right|_{J=0} = 0 = \langle \phi \rangle \qquad \frac{\delta^2 W}{\delta J \delta J} = \Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}\, 1}{k^2+m^2} = x_1 \text{———} x_2$

(81) $\left.\frac{\delta^{n>2} W}{\delta J_1 \ldots \delta J_n}\right|_{J=0} = 0 \Rightarrow$ only connected diagram is 2pt-function!

All connected $n>2$ vanish in free theory!

This immediately gives **Wick Theorem:**

(82) $$\boxed{\langle \phi_1 \ldots \phi_{2n} \rangle \underset{\text{def.}}{=} \sum_{\text{partition}} \prod \langle \ldots \rangle_c \underset{\text{Gaussian theory}}{=} \sum_{\text{partitions in pairs}} \prod_{\text{pairs}} \Delta_{ij}}$$

— Scalar Perturbation Theory —

(83) $$S[\phi] = \underbrace{\int d^4x \, \frac{1}{2} \phi(-\partial^2+m^2)\phi}_{S_0[\phi]} + S_{int}[\phi^{n\geq 3}, \partial]$$

(84) $$Z[\phi] = \int [d\phi] \exp(-S_0[\phi] - S_{int}[\phi] + J \cdot \phi)$$

This $Z[\phi]$ can be read in more ways, e.g.

(85) $$Z[\phi] = \int [d\phi] \exp(-S_0[\phi]) \exp(-S_{int}[\phi] + J \cdot \phi)$$

$$= \langle \exp(-S_{int}[\phi] + J \cdot \phi) \rangle_0 \cdot Z_0[0]$$

$Z_0[0] = \int [d\phi] \exp(-S_0)$

Gaussian, free & $J=0$
use Wick-Th.

$$\boxed{\text{Dyson Formula} \propto \langle 0 | T \exp(-S_{int}[\phi] + J \cdot \phi) | 0 \rangle_0}$$

can be calculated using Wick Theorem to any desired order:

sum of all diagrams built via Wick contractions using S_{int} & $J\phi$ as vertices.

Alternatively, think of $Z[J]$ as Fourier transform

$$(86) \quad Z[J] = \int [d\phi]\, \exp(-S_0 + J\cdot\phi)\, \exp(-S_{int}[\phi])$$

$$= \exp\left(-S_{int}\left[\tfrac{\delta}{\delta J}\right]\right) \int [d\phi]\, \exp(-S_0 + J\cdot\phi)$$

$$(87) \quad = \exp\left(-S_{int}\left[\tfrac{\delta}{\delta J}\right]\right) Z_0[J]$$

$$(88) \quad \boxed{Z[J] = \exp\left(-S_{int}\left[\tfrac{\delta}{\delta J}\right]\right) \exp\left(+\tfrac{1}{2} J\cdot\Delta\cdot J\right)}$$

Another explicit way of constructive $Z[J]$ in perturbation theory.

— 1PI Quantum Effective Action —

There is one more functional generator: the (quantum) effective action $\Gamma[\phi]$ that generates connected & 1PI diagrams.

1PI: "1-particle irreducible" = cannot be disconnected by cutting 1 line.

To motivate the definition, let's first look at the classical-only contribution when $\hbar \to 0$ in the path-integral.

$$(89) \quad Z[J] = \int [d\phi]\, \exp(-(S[\phi] - J\cdot\phi)/\hbar) \underset{\hbar\to 0}{\simeq} \exp(-(S[\bar\phi] - J\cdot\bar\phi)/\hbar)\,(1+\mathcal{O}(\hbar))$$

saddle at $\phi = \bar\phi(J)$

where we used the saddle-point approximation as crude estimate of integr.

$$(90) \quad \boxed{J \overset{\hbar\to 0}{=} \frac{\delta S}{\delta \phi}} \qquad \text{solved by} \quad \phi = \phi[J] \equiv \bar\phi$$

Since $Z[J] = e^{W[J]}$, we see that for $\hbar \to 0$

$$(91) \qquad \boxed{W[J] \underset{\hbar\to 0}{=} -S[\vartheta[J]] + J\cdot\vartheta[J]} \leftarrow \text{Legendre Transform}$$

That is, $W[J]$ for $\hbar \to 0$ is the Legendre transform of S.

One can even invert this relation solving for $\bar\vartheta = \vartheta[J]$

$$(92) \qquad J = J[\bar\vartheta] \Rightarrow \boxed{S[\bar\vartheta] \underset{\hbar\to 0}{=} -W[J[\bar\vartheta]] + J[\bar\vartheta]\cdot\bar\vartheta}$$

(where we recall that $J[\bar\vartheta]\cdot\bar\vartheta = \int d^4x \sum_i J_i[\bar\vartheta](x_i)\, \bar\vartheta_i(x_i)$)

This suggests to define an useful quantity via Legendre transf. also at $\hbar \neq 0$:

$$(93) \qquad W[J] = \log Z[J] \xrightarrow{\mathrm{I}} \boxed{\frac{\delta W}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z}{\delta J} = \langle\vartheta\rangle_J = \underset{\bar\vartheta}{\overset{\shortparallel}{\vartheta[J]}} \quad \text{inverting it} \quad J = J[\bar\vartheta]}$$

$\downarrow$ II Legendre Tr.

$$(94) \qquad \boxed{\begin{aligned} \Gamma[\bar\vartheta] &\equiv -W[J[\bar\vartheta]] + J[\bar\vartheta]\cdot\bar\vartheta \\ W[J] &= -\Gamma[\vartheta=\vartheta[J]] + J\cdot\vartheta[J] \end{aligned}} \quad \text{Definition of } \Gamma$$

Remarks:
- (31) is very useful already at $\hbar\to 0$ to calculate high multiplicity tree-diagrams
- Γ is functional of vacuum exp. values $\bar\vartheta = \langle\vartheta\rangle_J$ in J-bkg.
- $\langle\vartheta\rangle_{J=0}$ = extremum quantum action

$$(95) \qquad \frac{\delta\Gamma[\bar\vartheta]}{\delta\bar\vartheta} = \cancel{-\underset{\bar\vartheta}{\underset{\shortparallel}{\frac{\delta W}{\delta J}}}\frac{\delta J}{\delta\bar\vartheta}} + \cancel{\frac{\delta J}{\delta\bar\vartheta}\bar\vartheta} + J[\bar\vartheta] \Rightarrow \frac{\delta\Gamma}{\delta\bar\vartheta} = J$$

For $J=0$ $\quad \bar\vartheta = \langle\vartheta\rangle_{J=0}$ solves $\boxed{\delta\Gamma/\delta\bar\vartheta = 0}$ $(J=0)$

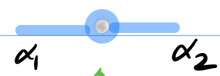
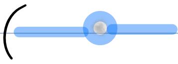
(96) $$\frac{\delta^2\Gamma}{\delta\bar\theta_1\delta\theta_2} = (\text{exact propagator})^{-1}_{\text{conn.}} = \left(\langle 0|T\theta_1\theta_2|0\rangle\right)^{-1}_{\text{conn.}}$$

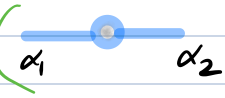
indeed, $$\frac{\delta^2\Gamma}{\delta\bar\theta_1\delta\bar\theta_2} = \frac{\delta J_1}{\delta\theta_2} = \left(\frac{\delta\bar\theta_2}{\delta J_1}\right)^{-1} = \left(\frac{\delta^2 W}{\delta J_2\delta J_1}\right)^{-1} = \left(\langle 0|T\theta_1\theta_2|0\rangle\right)^{-1}_{\text{conn.}}$$

$$\left(\int \frac{\delta J_1}{\delta\theta_2}\frac{\delta\bar\theta_2}{\delta J_3}dx_2 = \frac{\delta J_1}{\delta J_3} = \delta_{\alpha_1\alpha_3} = \delta_{i_1 i_3}\,\delta^4(x_1-x_3)\right)$$

Equivalently, $$\int d^4z\,\frac{\delta^2\Gamma}{\delta\bar\theta(x)\delta\bar\theta(z)}\,\frac{\delta^2 W}{\delta J(z)\delta J(y)} = \delta^4(x-y)$$

So $\Gamma^{(2)}_{ij} = \dfrac{\delta^2\Gamma}{\delta\bar\theta_i\delta\bar\theta_j}$ is the inverse of (connected) propagator $W^{(2)}_{ij}$

(97) $$\boxed{W^{(2)}_{\alpha_1\alpha_2} = \Gamma^{(2)\,-1}_{\alpha_1\alpha_2} =}$$  $\leftrightarrow$ ( $)^{-1}_{\alpha_1\alpha_2} = \Gamma^{(2)}_{\alpha_1\alpha_2}$

exact connected 2pt-function $\longrightarrow$ ( $\equiv$ )

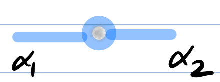
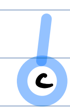
What about higher derivatives?

Defining

(98) $$\Gamma^{(n)} \equiv \Gamma_{\alpha_1\ldots\alpha_n} = \frac{\delta\Gamma}{\delta\bar\theta_{\alpha_1}\ldots\delta\theta_{\alpha_n}} \qquad W^{(n)} \equiv W_{\alpha_1\ldots\alpha_n} = \frac{\delta^{(n)} W}{\delta J_{\alpha_1}\ldots\delta J_{\alpha_n}}$$

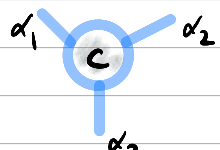
and recalling that

(99) $$\begin{cases} \dfrac{\delta}{\delta J}M^{-1} = -M^{-1}\dfrac{\delta M}{\delta J}M^{-1} & \left(M^{-1}M = \mathbb{I} \quad \dfrac{\delta M^{-1}}{\delta J}M + M^{-1}\dfrac{\delta M}{\delta J} = 0\right) \\[2ex] \dfrac{\delta}{\delta J_\alpha} = \dfrac{\delta\bar\theta_\beta}{\delta J_\alpha}\dfrac{\delta}{\delta\bar\theta_\beta} = W^{(2)}_{\beta\alpha}\dfrac{\delta}{\delta\bar\theta_\beta} & \end{cases}$$

(100)  $= W^{(2)}_{\alpha_1\alpha_2} = \Gamma^{2\;-1}_{\alpha_1\alpha_2} = \dfrac{\delta\theta_{\alpha_1}}{\delta J_{\alpha_2}} = \dfrac{\delta}{\delta J_{\alpha_2}}$ 

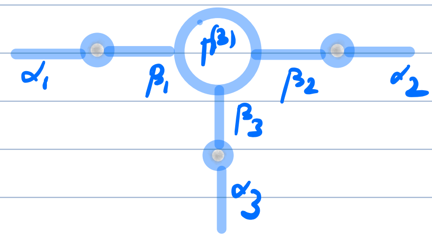
$\downarrow \dfrac{\delta}{\delta J_{\alpha_3}}$

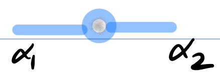
$\downarrow \frac{\delta}{\delta J_{\alpha_3}}$

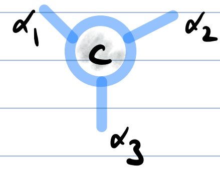
(101) 

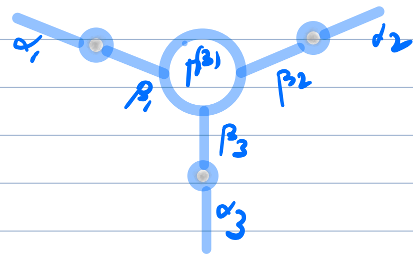
$$= W^{(3)}_{\alpha_1\alpha_2\alpha_3} = -\left(\Gamma^{(2)^{-1}}\right)_{\alpha_1\beta_1}\frac{\delta\,\Gamma^{(2)}_{\beta_1\beta_2}}{\delta J_{\alpha_3}}\left(\Gamma^{(2)^{-1}}\right)_{\beta_2\alpha_2}$$

$$= -W^{(2)}_{\alpha_1\beta_1}\,\Gamma^{(3)}_{\beta_1\beta_2\beta_3}\,W^{(2)}_{\beta_3\alpha_3}\,W^{(2)}_{\beta_2\alpha_2}$$

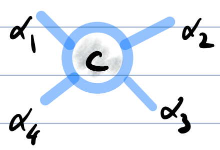
(102) $= -$ 

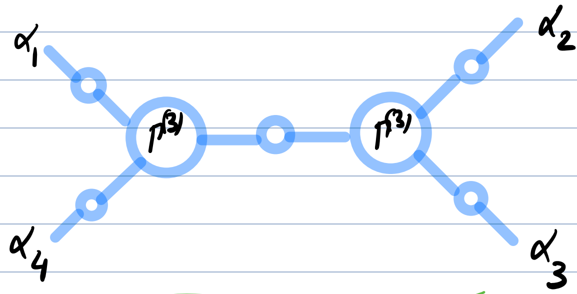
(103) $\frac{\delta}{\delta J_{\alpha_3}}$ 

$=$ 

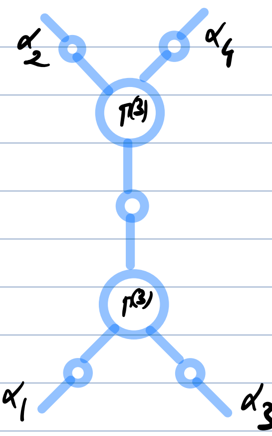
$= -$ 

$\downarrow \frac{\delta}{\delta J_{\alpha_4}}$

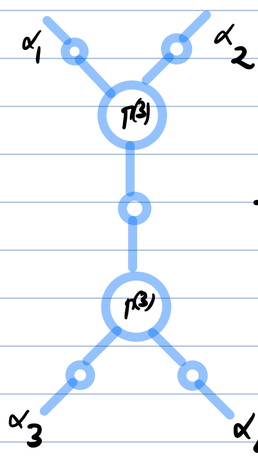
(104) 

$=$ 

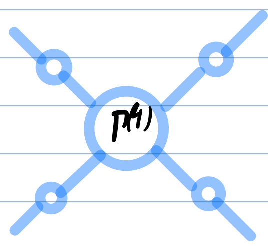
$\underbrace{\qquad}_{\frac{\delta}{\delta J_{\alpha_4}} W^{(2)}_{\alpha_1\beta_1}}$

$+$ 

$\underbrace{\qquad}_{\frac{\delta}{\delta J_{\alpha_4}} W^{(2)}_{\beta_2\alpha_2}}$

$+$ 

$\underbrace{\qquad}_{\frac{\delta}{\delta J_{\alpha_4}} W^{(2)}_{\beta_3\alpha_3}}$

$-$ 

$\underbrace{\qquad}$

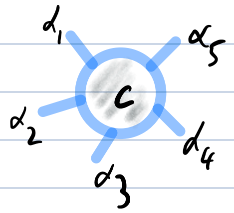
branches off new leg.

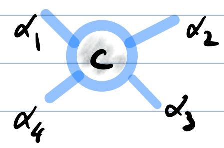
$$\frac{\delta}{\delta J_{\alpha_3}}\Gamma^{(3)}_{\beta_1\beta_2\beta_3} = \frac{\delta\bar\Theta_{\beta_4}}{\delta J_{\alpha_3}}\frac{\delta}{\delta\bar\Theta_{\beta_4}}\Gamma^{(3)}_{\beta_1\beta_2\beta_3} = W^{(2)}_{\alpha_3\beta_4}\,\Gamma^{(4)}_{\beta_1\beta_2\beta_3\beta_4}$$

In other words, the rules are

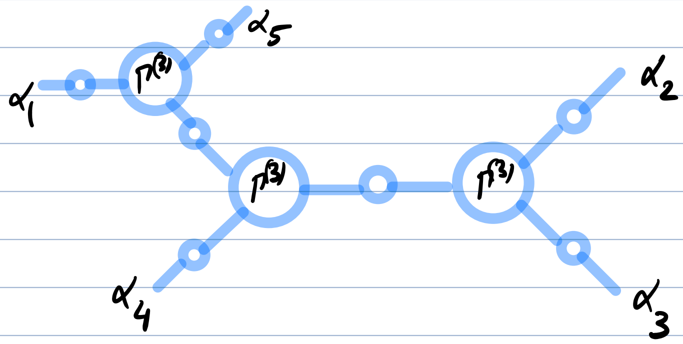
(105)
$$\boxed{\begin{aligned} \frac{\delta}{\delta J_{\alpha_i}} \Gamma^{(n)}_{\alpha_1\dots\alpha_n} &= \Gamma^{(n+1)}_{\alpha_1\dots\alpha_n\beta}\, W^{(2)}_{\beta\alpha_i} \qquad n \geq 3 \\ \frac{\delta}{\delta J_{\alpha_i}} W^{(2)}_{\alpha_1\alpha_2} &= W^{(3)}_{\alpha_1\alpha_2\alpha_3} = - W^{(2)}_{\alpha_1\beta_1} W^{(2)}_{\alpha_2\beta_2} W^{(2)}_{\alpha_i\beta_3} \Gamma^{(3)}_{\beta_1\beta_2\beta_3} \end{aligned}}$$

One can clearly keep going to higher-point correlators & express all of them in terms of $W^{(2)} (= \Gamma^{(2)^{-1}})$ & $\Gamma^{(n)}$:

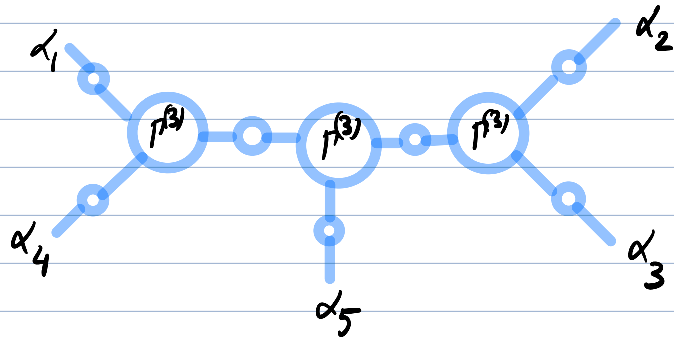
(106) 

$= \frac{\delta}{\delta J_{\alpha_5}}$ 

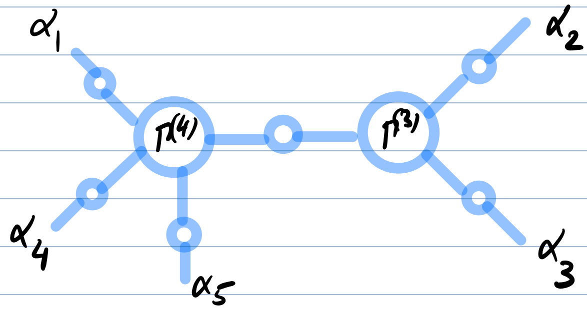
$= \frac{\delta}{\delta J_{\alpha_5}}$ (104)

$= -$ 

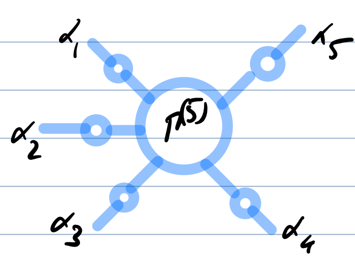
$+$ permut. external legs

$-$ 

$+$ permut. ext. legs

$+$ 

$+$ permutations

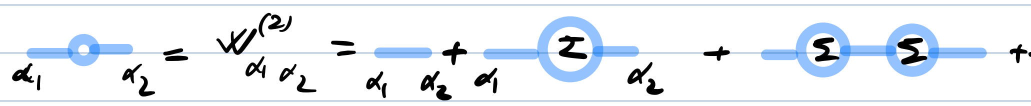
$+$ 

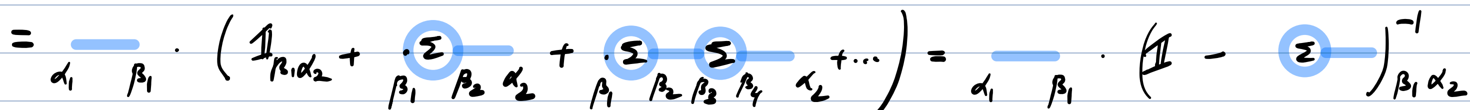
Remarks:

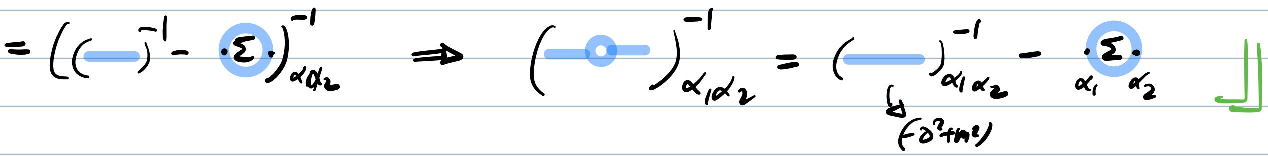
(a) $\Gamma^{(2)} = W^{(2)\,-1}$ generalize the notion of kinetic term

$\left(W^{(2)-1}\big|_{free} = \delta(x-y)(-\partial^2 + m^2) \right)$

This can be seen also in perturbation theory by resumming all connected $n=2$-1PI diagrams Σ :

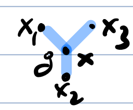


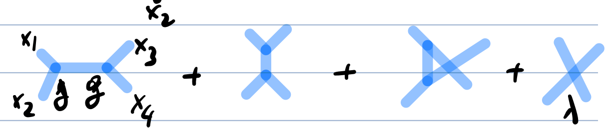




(b) $\Gamma^{(n)}$: generalizes the notion of vertex (where to attach propagators to build correlators)

Example: $\begin{cases} W^{(n=2)}\big|_{gaussian} = \bullet\!\!-\!\!\bullet = \Delta \\ W^{n\geq 3}\big|_{gaussian} = 0 \end{cases}$ $\Rightarrow$ $W^{(n=3)}\Big|_{V=\frac{g}{3!}\phi^3\text{-theory}} = \int d^4x \; -\lambda \, \Delta_{xx_1} \Delta_{xx_2} \Delta_{xx_3} + O(g^2)$

$=$  $+ \dots$

$W^{(n=4)}\Big|_{\frac{g\phi^3}{3!}+\frac{\lambda\phi^4}{4!}} =$ 

Same type of structures made of propagator & vertices at tree-level, except that the vertices are $\Gamma^{(n)}$, the propagator exact & the tree-level (1PI-diagrams) are exact!

Let's show this by considering $W_\Gamma[J]$ to be the partition function that we would get from path integral under replacement

$S[\phi] \longrightarrow \Gamma[\phi]$, and restoring $\hbar$:

(107) $W_\Gamma[J] = \log \int [D\phi] \exp\left(-\left(\Gamma[\phi] - J\cdot\phi\right)/\hbar\right)$

Imagine doing perturbation theory the new "free" propagator would be $(\Gamma^{(2)})^{-1}\hbar = W^{(2)}\hbar$, and each vertex would be $\Gamma^{(n)}/\hbar$

(108) $\boxed{\text{connected diagram w/ } I \text{ internal lines \& } V \text{ vertices} \propto \hbar^{I-V} = \hbar^{L-1}}$

$\#\text{ loop} = \#\text{ indip. mom. assign.} \longrightarrow L = I - V + 1$

when $\hbar \to 0$ only the tree-diagrams survive. On the other hand, by construction, when $\hbar \to 0$ $\underline{W_\Gamma[J] \overset{\hbar\to 0}{=} -\left(\Gamma[\bar\phi] - J\bar\phi\right)/\hbar}$ which is the Legendre inverse of Γ, that is $W[J]$ (see Eq. (94))

(108) $\boxed{W_\Gamma^{tree} = W[J]} = \int_{\substack{\text{connected}\\ \text{Tree-only}}} [D\phi] \exp\left(-\left(\Gamma[\phi] - J\cdot\phi\right)\right)$ (restoring $\hbar=1$)

This is just putting in a closed formula the diagrams that we were getting (104-106) which are indeed all tree, no loop with exact propagator, and yet capture exactly all connected correlators.

Comment: This functional is called 1PI because the exact correlators are sum of terms that becomes disconnected if cutting 1- line + a reminder that then can't be disconnected in that way. another reason is the following path integral representation of Γ

— Path-integral representation of Γ —

We can give a path-integral representation of Γ from the definition (94)

$$(110)\qquad e^{-\Gamma[\bar\vartheta]+J[\bar\vartheta]\cdot\bar\vartheta} = e^{W[J[\bar\vartheta]]} = \int[d\vartheta]\exp\left(-S[\vartheta]+J[\bar\vartheta]\cdot\vartheta\right)$$

$$\Downarrow$$

$$(111)\qquad e^{-\Gamma[\bar\vartheta]} = \int[d\vartheta]\exp\left(-S[\vartheta]+J[\bar\vartheta]\cdot(\vartheta-\bar\vartheta)\right) \underset{\vartheta-\bar\vartheta=\vartheta'}{=} \int[d\vartheta']\exp\left(-S[\vartheta'+\bar\vartheta]+J[\bar\vartheta]\cdot\vartheta'\right)$$

$\langle\vartheta'\rangle_{J[\bar\vartheta]}=0$

where $J=J[\bar\vartheta]$ is such that $\langle\vartheta\rangle_J=\bar\vartheta$

that is $\langle\vartheta'\rangle_J=0$ so that any ϑ'-tadpole must be vanishing, hence (111) being equivalent to

$$(112)\qquad \boxed{e^{-\Gamma[\bar\vartheta]} = \int_{\substack{\text{1PI}\\ \text{connect.}}}[D\vartheta']\exp\left(-S[\vartheta'+\bar\vartheta]\right)}$$

(← Known as bkg field method)

The restriction to 1PI is what removes single ϑ-legs, i.e. it removes 1PI irreducible diagrams, those that become disconnected by cutting one internal line only.

— Momentum-Space —

We have been quite formal with the indices so far, so that one can easily change basis, e.g. go to momentum-space correlation.

For example, for a scalar field in k-space:

$$(113)\qquad W^{(2)}(k_1,k_2) = \langle 0|T\,\hat\phi(k_1)\,\hat\phi(k_2)|0\rangle_{\text{conn.}} = \left(\Gamma^{(2)}(k_1,k_2)\right)^{-1} = (2\pi)^4\delta^4(k_1+k_2)\,\tilde W^{(2)}(k_i)$$

↑mom. indices now

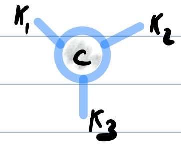
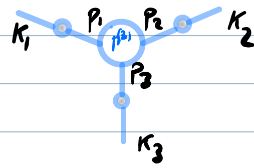
↑translation inv.

Example: Free theory: $W^{(2)}(k_1,k_2) = \dfrac{(2\pi)^4\delta^4(k_1+k_2)}{k^2+m^2}$ (euclidean)

$$(114) \quad W^{(n)}(k_1,\dots,k_n) = \langle 0|T\,\hat\phi(k_1)\dots\hat\phi(k_n)|0\rangle \underset{\text{transl. inv.}}{=} (2\pi)^4\delta^4(\Sigma_i k_i)\,\tilde W^{(n)}(k_i)$$

Analogously, we define $\tilde\Gamma^{(n)}$ by factoring out $(2\pi)^4\delta^4(\Sigma k_i)$, $\Gamma^{(n)} = (2\pi)^4\delta^4(\ )\,\tilde\Gamma^{(n)}$

Example:

$$(115) \quad W^{(3)}(k_1,k_2,k_3) =$$  $= -$ 

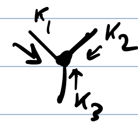
$\propto \delta^4(k_1+k_2+k_3)$

$$= -\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)}\frac{d^4p_3}{(2\pi)^4}\, W^{(2)}(k_1,p_1)\, W^{(2)}(k_2,p_2)\, W^{(2)}(k_3,p_3)\, \Gamma^{(3)}(p_1,p_2,p_3)$$

$W^{(2)}(k_1,p_1)$: $\propto \delta^4(k_1+p_1)$; $\Gamma^{(3)}(p_1,p_2,p_3)$: $\delta^4(p_1+p_2+p_3)$

$$(116) \quad \tilde W^{(3)}(k_1,k_2,k_3) = -\tilde W^{(2)}(k_1)\,\tilde W^{(2)}(k_2)\,\tilde W^{(2)}(k_3)\,\tilde\Gamma^{(3)}(k_1,k_2,k_3)$$

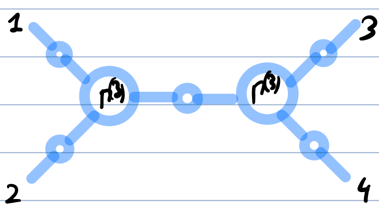
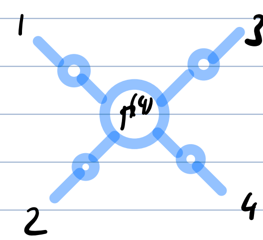
$$\underset{\substack{\text{in pert. theory}\\ V = g\,\phi^3/3!}}{=} -\frac{1}{k_1^2+m^2}\,\frac{1}{k_2^2+m^2}\,\frac{1}{k_3^2+m^2}\cdot g + o(g^3))$$



The momentum space (connected) correlators are important also because we can use them to extract (connected) Scattering Amplitudes via LSZ.

$$\boxed{\mathrm{LSZ}\cdot \tilde W^{(n)}\Big|_{\text{EUC} \xrightarrow{\text{W.R.}} \text{MINK}} = \text{Scattering Amplitudes}}$$

What is the connection with $\tilde\Gamma^{(n)}$?

$$(117) \quad W^{(4)}_{\text{EUC}} = \Big($$  $+$ permut. $\Big) -$ 

$$(118) \quad W^{(n)}_{\text{EUC}}(x_1,\dots,x_n) = \int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_n}{(2\pi)^4}\, e^{-ip_1x_1}\cdots e^{-ip_nx_n}(2\pi)^4\delta^4(\Sigma_i p_i)\,\tilde W^{(n)}_{\text{EUC}}(p_i)$$

Wick-Rotation = $\begin{cases} \tau \longrightarrow i\,t(1-i\varepsilon) = i\,x^0(1-i\varepsilon) \\ p_0 \longrightarrow -i\,p_0(1+i\varepsilon) = -i\,p_0(1+i\varepsilon) \end{cases}$

$\Downarrow$

(119) $$p\cdot x\big|_{euc} = \tau p_0 + \vec{x}\,\vec{p} \longrightarrow x^0 p_0 + x^i p_i = x^\mu p_\mu\big|_{Mink}$$

There are $(n-1)$-integrations $\Rightarrow$ factor $(-i)^{n-1}$

(120) $$W^{(n)}_{Mink}(x_1,\dots,x_n) = (-i)^{n-1}\int \underbrace{\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_{n-1}}{(2\pi)^4}}_{Mink}\, e^{-i\overbrace{p_1(x_1-x_n)}^{Mink}}\cdots e^{-ip_{n-1}(x_{n-1}-x_n)}\,\tilde{W}^{(n)}\big|_{Euc}(-ip_0^i)$$

$\Downarrow$

(121) $$\boxed{\tilde{W}^{(n)}_{Mink}(p_\mu^{Mink}) = (-i)^{n-1}\,\tilde{W}_{Euc}^{(n)}(p_{0}^{eu} \to -ip_0)}$$

Example: $$\tilde{W}^{(2)}\big|_{Euc}\Big|_{Free} = \frac{1}{p_{Euc}^2+m^2} \xrightarrow{\text{Wick}} \frac{-i}{-p_0^2(1+i\varepsilon)^2+\vec{p}^{\,2}+m^2} = \frac{i}{p^2-m^2+i\varepsilon}$$

Since $\tilde{W}^{(n)}\big|_{euc} \supset \left(W^{(2)}_{Euc}\right)^n$ <u>From external legs</u>

$\Downarrow$

(122) $$\tilde{W}^{(n)}\big|_{Mink} \supset (-i)^{n-1}\left(W^{(2)}\big|_{euc}\right)^n(p^0_{eu}\to -ip^0) = +i\left(W_{Mink}(p)\right)^n$$

Moreover, LSZ multiply $\tilde{W}^{(n)}_{Mink}$ by $\prod_{i=1}^{n}\left(\frac{p_i^2-m_i^2}{i|Z_i|}\right)$

(123) $$LSZ\cdot\tilde{W}^{(n)}\big|_{Mink}(p) \supset i\left[\prod_{i=1}^{n}\frac{(p_i^2-m_i^2)}{i|Z_i|}\,W^{(2)}_{Mink}(p)\right]$$

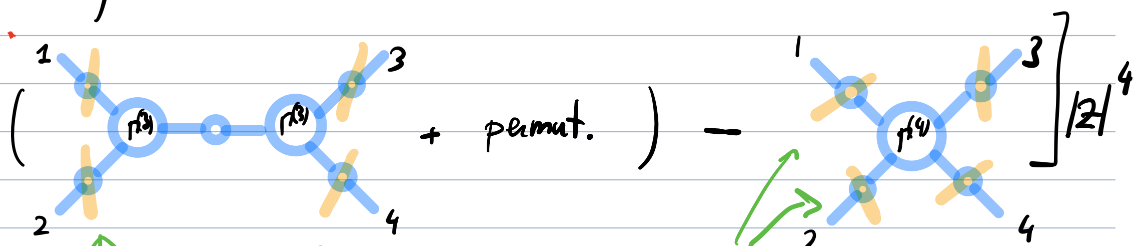
On the other hand, we know from Källen-Lehmen that

(124) $$\tilde{W}^{(2)}_{Mink)} = \int d\mu^2\,\rho(\mu^2)\,\frac{i}{p^2-m^2+i\varepsilon}$$ so that (123) $\longrightarrow$ on-shell

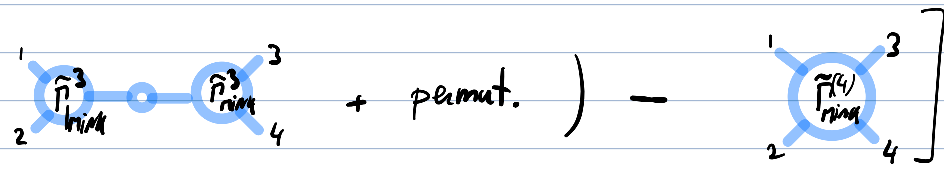
does not vanish only if

(125) $\rho(\mu^2)$ $\quad \delta(\mu^2-m_i^2)\,|z_i|^2$ ← residue at pole > 0

Therefore, the pole is removed, the residue $|z_i|^2/|z_i| = |z_i|$, and extra factor of "i" in (123)

(126) $LSZ\cdot \tilde{W}^{(4)}_{Mink} = i\Big[$  $\Big]\,|z|^4$

amputated

(127) $= i\Big[$  $\Big]\cdot |z|^4$

(128) $= i\, M_{2\to 2}$ $\qquad$ (from $S = \mathbb{1} + iM(2\pi)^4\delta^4(\Sigma_i p_i)$)

(129) $\boxed{M_{2\to 2} = \left(\prod \sqrt{\text{residues}}\right)\cdot \text{ampu\-tated Wick-rotated connected Correlators}}$

which are conveniently built out of $\Gamma^{(n)}$ & internal lines.

Example: Take theory with $\phi \to -\phi$ symmetry $\Rightarrow \Gamma^{(2n+1)} = 0$

e.g. ϕ^4-theory. Take $V = \frac{\lambda}{4!}\phi^4$

$$\tilde{W}^{(4)}\Big|_{\substack{\text{amputated}\\ \text{Mink}}} = -i\,\tilde{\Gamma}^{(4)}_{Mink} = -i\,\tilde{\Gamma}^{(4)}_{Euc}\big(p^0 \to -ip^0(1+i\varepsilon)\big) = i\,M_{2\to 2}$$

$z = 1 + O(\lambda^2)$ $\qquad M_{2\to2} = i\lambda + O(\lambda^2)$

Aside: $W[J]$ & the Replica Trick

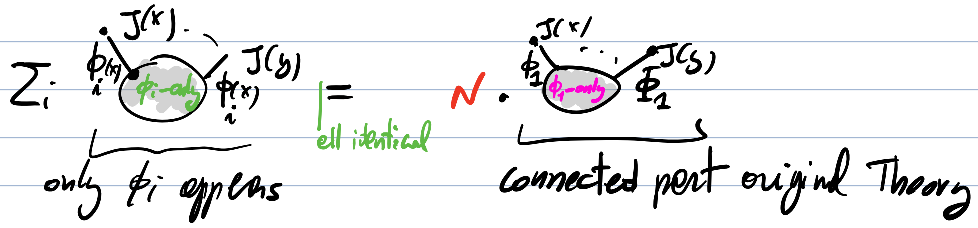
We can obtain again that $W[J]$ generates the connected diagrams using the Replica Trick from statistical Mech. Consider N copies of the system which share the same current but have no interactions among themselves (each it's interacting on its own)

$$(130)\quad Z_N[J] = \int[D\phi_1]\dots[D\phi_N]\, e^{-(S[\phi_1]+S[\phi_2]+\dots S[\phi_N])-J(\phi_1+\dots\phi_N)}$$

$$= (Z[J])^N. \qquad (Z[J]:\text{ original 1-copy partition function})$$

A single connected diagram is now proportional to N because J couples to all $\phi_{i=1,\dots,N}$ but they don't talk to each other so that any connected diagram is replicated N times

(131)



(If there are n disconnected diagrams, it would contribute N^n since we can choose between N fields in each disconnected diagram)

Therefore, in order to extract the connected terms of the original theory is enough to look at $O(N)$, which we can read directly from Eq. (68)

$$(132)\quad Z_N[J] = e^{N\ln Z[J]} = 1 + N\ln Z[J] + O(N^2) \Rightarrow \boxed{W[J] = \ln Z[J]}$$

sum of connected diag.